

ON HERMITE-HADAMARD INEQUALITIES FOR DIFFERENTIABLE λ -PREINVEX FUNCTIONS VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, we consider a new class of convex functions which is called λ -preinvex functions. We prove several Hermite–Hadamard type inequalities for differentiable λ -preinvex functions via Fractional Integrals. Some special cases are also discussed.

1. INTRODUCTION

The convexity property of a given function plays an important role in obtaining integral inequalities. Proving inequalities for convex functions has a long and rich history in mathematics. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality:

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

is known in the literature as Hermite-Hadamard inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f . Both inequalities hold in the reversed direction if f is concave.

Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [1]-[20]) and the references therein.

A significant generalization of convex functions is that of invex functions introduced by Hanson in [9]. Ben-Israel and Mond [3] introduced the concept of preinvex functions, which is a special case of invexity. Noor [10]-[13] has established some Hermite-Hadamard type inequalities for preinvex and logpreinvex functions. In recent papers Barani, Ghazanfari, and Dragomir in [4] presented some estimates of the right hand side of a Hermite- Hadamard type inequality in which some preinvex functions are involved. His class of nonconvex functions include the classical convex functions and its various classes as special cases. For some recent results related to this nonconvex functions, see the papers ([10]-[13], [14]).

Now, we will give some definitions, lemmas and notations which we use later in this work.

Key words and phrases. Fractional Hermite-Hadamard inequalities, preinvex functions, Riemann-Liouville Fractional Integral.

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Definition 1. ([15]) Let $f \in L[a, b]$. The Riemann-Liouville fractional integral $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a > 0$ are defined by

$$(1.2) \quad \begin{aligned} J_{a+}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad 0 \leq a < x \leq b \\ J_{b-}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad 0 \leq a < x \leq b \end{aligned}$$

Where Γ is the gamma function.

Definition 2. ([6]) The incomplete beta function is defined as follows:

$$(1.3) \quad B_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt,$$

Here $x \in [0, 1]$, $a, b > 0$.

Definition 3. ([2]) Gaussian hypergeometric function defined by

$$(1.4) \quad {}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt$$

Here $c > b > 0$, $|z| < 1$.

Definition 4. ([19]) A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to belong to the class $MT(I)$ if f is positive and $\forall x, y \in I$ and $t \in (0, 1)$ satisfies the inequality:

$$(1.5) \quad f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y).$$

Definition 5. A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to a λ - MT -convex function or said to belong to the class λ - $MT(I)$ if f is positive and $\forall x, y \in I$, $\lambda \in (0, \frac{1}{2}]$ and $t \in (0, 1)$ satisfies the inequality:

$$(1.6) \quad f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(y).$$

Meanwhile, Sarikaya et al. [16] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard-type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha \in \mathbb{R}^+$.

Lemma 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a once differentiable mapping on (a, b) for $a < b$. If $f' \in L[a, b]$, there is a following equality for fractional integrals

$$(1.7) \quad \begin{aligned} &\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \\ &= \frac{b-a}{2} \int_0^1 [(1-t)^{\alpha} - t^{\alpha}] f'(ta + (1-t)b) dt. \end{aligned}$$

Also, Wang et al. [20] presented the following inequality.

Lemma 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable mapping on (a, b) for $a < b$. If $f'' \in L[a, b]$, there is following equality for fractional integrals

$$(1.8) \quad \begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \\ &= \frac{(b-a)^2}{2} \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] f''(ta + (1-t)b) dt. \end{aligned}$$

In [5], Dragomir and Agarwal established the following result connected with the right part of (1.1):

Theorem 1. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:

$$(1.9) \quad \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|].$$

Lemma 3. ([18]) For any $A_1 > A_2 \geq 0$ and $p \geq 1$, $(A_1 - A_2)^P \leq A_1^P - A_2^P$.

Lemma 4. ([7]) For $t \in [0, 1]$, we have

$$(1.10) \quad \begin{aligned} (1-t)^m &\leq 2^{1-m} - t^m && \text{for } m \in [0, 1], \\ (1-t)^m &\geq 2^{1-m} - t^m && \text{for } m \in [1, \infty). \end{aligned}$$

Let \mathbb{R}^n be Euclidian space and K is said to a nonempty closed in \mathbb{R}^n . Let $f : K \rightarrow \mathbb{R}$ and $\eta : K \times K \rightarrow \mathbb{R}$ be a continuous functions.

Definition 6. ([10]) Let $u \in K$. The set K is said to be invex at u according to η if

$$(1.11) \quad u + t\eta(v, u) \in K$$

for all $u, v \in K$ and $t \in [0, 1]$.

Definition 7. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative function. A function f on the set K_η is said to be λ -preinvex function according to bifunction η and $\forall u, v \in I$, $t \in (0, 1)$, then

$$(1.12) \quad f(u + t\eta(v, u)) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(u).$$

Remark 1. In Definition 7, if we choose $\lambda = \frac{1}{2}$, and $\eta(v, u) = v - u$. Definition 7 reduces to Definition 4;

$$f(tv + (1-t)u) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(u).$$

Remark 2. In Definition 7, if we choose $\eta(v, u) = v - u$. Definition 7 reduces to Definition 5;

$$f(tv + (1-t)u) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(u).$$

Our goal in this paper is to state and prove the Hermite-Hadamard type inequality for preinvex functions via Riemann-Liouville Fractional Integrals. In order to achieve our goal, we first give two important lemmas and then by using these identities we prove some integral inequalities.

2. MAIN RESULTS

We need the following lemma [8].

Lemma 5. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. If $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L[a, a + \eta(b, a)]$ then, the following equality holds:*

$$(2.1) \quad \begin{aligned} & \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} \left[J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a + \eta(b, a))^-}^\alpha f(a) \right] \\ &= \frac{\eta(b, a)}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(a + (1-t)\eta(b, a)) dt. \end{aligned}$$

Proof. Integrating by part and changing the variable of definite integral yield

$$(2.2) \quad \begin{aligned} & \int_0^1 [(1-t)^\alpha - t^\alpha] f'(a + (1-t)\eta(b, a)) dt \\ &= [(1-t)^\alpha - t^\alpha] \frac{f(a + (1-t)\eta(b, a))}{-\eta(b, a)} \Big|_0^1 \\ & \quad - \frac{\alpha}{\eta(b, a)} \int_0^1 [(1-t)^{\alpha-1} + t^{\alpha-1}] f(a + (1-t)\eta(b, a)) dt \\ &= \frac{f(a) + f(a + \eta(b, a))}{\eta(b, a)} - \frac{\alpha}{\eta(b, a)} \left[\frac{1}{(\eta(b, a))^\alpha} \int_a^{a + \eta(b, a)} (a + \eta(b, a) - x)^{\alpha-1} f(x) dx \right. \\ & \quad \left. + \frac{1}{(\eta(b, a))^\alpha} \int_a^{a + \eta(b, a)} (x - a)^{\alpha-1} f(x) dx \right] \\ &= \frac{f(a) + f(a + \eta(b, a))}{\eta(b, a)} - \frac{\Gamma(\alpha + 1)}{(\eta(b, a))^{\alpha+1}} \left[J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a + \eta(b, a))^-}^\alpha f(a) \right]. \end{aligned}$$

By multiplying the both sides of (2.2) by $\frac{\eta(b, a)}{2}$, we have:

$$\begin{aligned} & \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} \left[J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a + \eta(b, a))^-}^\alpha f(a) \right] \\ &= \frac{\eta(b, a)}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(a + (1-t)\eta(b, a)) dt. \end{aligned}$$

Lemma 5 is thus proved. \square

Remark 3. *In Lemma 5, if we choose $\eta(b, a) = b - a$, Lemma 5 reduces to Lemma 1;*

Theorem 2. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L[a, a + \eta(b, a)]$. If $|f'|$ is λ -preinvex function on $[a, a + \eta(b, a)]$ then*

the following inequality for fractional integrals with $\alpha > 0$ holds:

$$\begin{aligned}
& \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} \left[J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a + \eta(b, a))^-}^\alpha f(a) \right] \right| \\
& \leq \frac{\eta(b, a)}{8} \left[|f'(a)| + \frac{1 - \lambda}{\lambda} |f'(b)| \right] \left\{ \frac{2\sqrt{\pi}\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha + 2)} - \frac{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 2)} - 4B_{\frac{1}{2}}(\alpha + \frac{3}{2}, \frac{1}{2}) \right. \\
& \quad + \frac{2^{-\alpha}(- (4\alpha^2 + 18\alpha + 19) {}_2F_1(1, \alpha + 2; \frac{1}{2}; \frac{1}{2}) - 2(\alpha + 2) {}_2F_1(1, \alpha + 2; -\frac{1}{2}; \frac{1}{2}))}{4\alpha^2 + 8\alpha + 3} \\
& \quad \left. + \frac{2^{-\alpha}(-\alpha + 2^{\alpha+\frac{1}{2}} {}_2F_1(-\frac{1}{2}, \frac{1}{2} - \alpha; \frac{1}{2}; \frac{1}{2}) - 1)}{\alpha(\alpha + 1)} \right\}.
\end{aligned}$$

Proof. By using Definition 7 and Lemma 5, we have:

$$\begin{aligned}
& \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} \left[J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a + \eta(b, a))^-}^\alpha f(a) \right] \right| \\
& \leq \frac{\eta(b, a)}{2} \int_0^1 |(1 - t)^\alpha - t^\alpha| |f'(a + (1 - t)\eta(b, a))| dt \\
& \leq \frac{\eta(b, a)}{2} \left[\int_0^{\frac{1}{2}} [(1 - t)^\alpha - t^\alpha] |f'(a + (1 - t)\eta(b, a))| dt \right. \\
& \quad \left. + \int_{\frac{1}{2}}^1 [t^\alpha - (1 - t)^\alpha] |f'(a + (1 - t)\eta(b, a))| dt \right] \\
& \leq \frac{\eta(b, a)}{2} \left[\int_0^{\frac{1}{2}} [(1 - t)^\alpha - t^\alpha] \left(\frac{\sqrt{t}}{2\sqrt{1-t}} |f'(a)| + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f'(b)| \right) dt \right. \\
& \quad \left. + \int_{\frac{1}{2}}^1 [t^\alpha - (1 - t)^\alpha] \left(\frac{\sqrt{t}}{2\sqrt{1-t}} |f'(a)| + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f'(b)| \right) dt \right] \\
& \leq \frac{\eta(b, a)}{2} \left[\int_0^1 [(1 - t)^\alpha + t^\alpha] \left(\frac{\sqrt{t}}{2\sqrt{1-t}} |f'(a)| + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f'(b)| \right) dt \right] \\
& \leq \frac{\eta(b, a)}{8} \left[|f'(a)| + \frac{1 - \lambda}{\lambda} |f'(b)| \right] \left\{ \frac{2\sqrt{\pi}\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha + 2)} - \frac{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 2)} - 4B_{\frac{1}{2}}(\alpha + \frac{3}{2}, \frac{1}{2}) \right. \\
& \quad + \frac{2^{-\alpha}(- (4\alpha^2 + 18\alpha + 19) {}_2F_1(1, \alpha + 2; \frac{1}{2}; \frac{1}{2}) - 2(\alpha + 2) {}_2F_1(1, \alpha + 2; -\frac{1}{2}; \frac{1}{2}))}{4\alpha^2 + 8\alpha + 3} \\
& \quad \left. + \frac{2^{-\alpha}(-\alpha + 2^{\alpha+\frac{1}{2}} {}_2F_1(-\frac{1}{2}, \frac{1}{2} - \alpha; \frac{1}{2}; \frac{1}{2}) - 1)}{\alpha(\alpha + 1)} \right\}.
\end{aligned}$$

The proof is done. \square

Remark 4. If we take $\eta(b, a) = b - a$, $\lambda = \frac{1}{2}$ and $\alpha = 1$ in Theorem 2, Theorem 2 reduces to Theorem 1;

Theorem 3. Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L[a, a + \eta(b, a)]$. If $|f'|^q$ is λ -preinvex function on $[a, a + \eta(b, a)]$ for some fixed $q > 1$ then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} \left[J_{a+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b, a))^-}^\alpha f(a) \right] \right| \\ & \leq \frac{\eta(b, a)}{2} \left(\frac{\pi}{4} \right)^{\frac{1}{q}} \left(\frac{2 - 2^{1-\alpha p}}{p\alpha + 1} \right)^{\frac{1}{p}} \left[|f'(a)|^q + \left(\frac{1-\lambda}{\lambda} \right) |f'(b)|^q \right]^{\frac{1}{q}} \end{aligned}$$

where $\alpha \in [0, 1]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Definition 7, Lemma 5 and Hölder's inequality, we have:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} \left[J_{a+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b, a))^-}^\alpha f(a) \right] \right| \\ & \leq \frac{\eta(b, a)}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(a + (1-t)\eta(b, a))| dt \\ & \leq \frac{\eta(b, a)}{2} \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(a + (1-t)\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{\eta(b, a)}{2} \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 \left(\frac{\sqrt{t}}{2\sqrt{1-t}} |f'(a)|^q + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f'(b)|^q \right) dt \right)^{\frac{1}{q}} \\ & \leq \frac{\eta(b, a)}{2} \left[\frac{\pi}{4} |f'(a)|^q + \frac{\pi}{4} \left(\frac{1-\lambda}{\lambda} \right) |f'(b)|^q \right]^{\frac{1}{q}} \\ & \quad \times \left(\int_0^{\frac{1}{2}} [(1-t)^{\alpha p} - t^{\alpha p}] dt + \int_{\frac{1}{2}}^1 [t^{\alpha p} - (1-t)^{\alpha p}] dt \right)^{\frac{1}{p}} \\ & \leq \frac{\eta(b, a)}{2} [|f'(a)|^q + \frac{1-\lambda}{\lambda} |f'(b)|^q]^{\frac{1}{q}} \left(\frac{\pi}{4} \right)^{\frac{1}{q}} \left(\frac{2 - 2^{1-\alpha p}}{\alpha p + 1} \right)^{\frac{1}{p}}. \end{aligned}$$

The proof is done. \square

Remark 5. In Theorem 3, if we choose $\eta(b, a) = b - a$, then we have;

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\ & \leq \frac{b-a}{2} \left(\frac{\pi}{4} \right)^{\frac{1}{q}} \left(\frac{2 - 2^{1-\alpha p}}{p\alpha + 1} \right)^{\frac{1}{p}} \left[|f'(a)|^q + \left(\frac{1-\lambda}{\lambda} \right) |f'(b)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

Remark 6. In Theorem 3, if we choose $\eta(b, a) = b - a$ and $\alpha = 1$, then we have;

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \left[\frac{\pi}{4} |f'(a)|^q + \frac{\pi}{4} \left(\frac{1-\lambda}{\lambda} \right) |f'(b)|^q \right]^{\frac{1}{q}} \left(\frac{2-2^{1-p}}{p+1} \right)^{\frac{1}{p}}. \end{aligned}$$

Remark 7. In Theorem 3, if we choose $\eta(b, a) = b - a$, $\lambda = 1/2$ and $\alpha = 1$, then we have;

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{8} \pi [|f'(a)|^q + |f'(b)|^q]^{\frac{1}{q}} \left(\frac{2-2^{1-p}}{p+1} \right)^{\frac{1}{p}}. \end{aligned}$$

Theorem 4. Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L[a, a + \eta(b, a)]$. If $|f'|^q$ is λ -preinvex function on $[a, a + \eta(b, a)]$ for some fixed $q > 1$ then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} [J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a + \eta(b, a))^-}^\alpha f(a)] \right| \\ & \leq \left(\frac{1-2^{-\alpha}}{\alpha+1} \right)^{\frac{q-1}{q}} \frac{\eta(b, a)}{2^{1+1/q}} \left[|f'(a)|^q + \frac{1-\lambda}{\lambda} |f'(b)|^q \right] \\ & \times \left\{ \frac{2\sqrt{\pi}\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha + 2)} - \frac{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 2)} - 4B_{\frac{1}{2}}(\alpha + \frac{3}{2}, \frac{1}{2}) \right. \\ & + \frac{2^{-\alpha}(- (4\alpha^2 + 18\alpha + 19) {}_2F_1(1, \alpha + 2; \frac{1}{2}; \frac{1}{2}) - 2(\alpha + 2) {}_2F_1(1, \alpha + 2; -\frac{1}{2}; \frac{1}{2}))}{4\alpha^2 + 8\alpha + 3} \\ & \left. + \frac{2^{-\alpha}(-\alpha + 2^{\alpha+\frac{1}{2}} {}_2F_1(-\frac{1}{2}, \frac{1}{2} - \alpha; \frac{1}{2}; \frac{1}{2}) - 1)}{\alpha(\alpha + 1)} \right\}^{1/q}. \end{aligned}$$

where $\alpha \in [0, 1]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Definition 7, Lemma 5 and Power Mean inequality, we have:

$$\begin{aligned}
& \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} \left[J_{a+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b, a))^-}^\alpha f(a) \right] \right| \\
& \leq \frac{\eta(b, a)}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(a + (1-t)\eta(b, a))| dt \\
& \leq \frac{\eta(b, a)}{2} \left(\int_0^1 |(1-t)^\alpha - t^\alpha| dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 |(1-t)^\alpha - t^\alpha| |f'(a + (1-t)\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{\eta(b, a)}{2} \left(\int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 |(1-t)^\alpha - t^\alpha| |f'(a + (1-t)\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{\eta(b, a)}{2} \left(\frac{2-2^{1-\alpha}}{\alpha+1} \right)^{\frac{q-1}{q}} \left[\int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] \left(\frac{\sqrt{t}}{2\sqrt{1-t}} |f'(a)|^q + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f'(b)|^q \right) dt \right. \\
& \quad \left. + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] \left(\frac{\sqrt{t}}{2\sqrt{1-t}} |f'(a)|^q + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f'(b)|^q \right) dt \right]^{\frac{1}{q}} \\
& \leq \left(\frac{1-2^{-\alpha}}{\alpha+1} \right)^{\frac{q-1}{q}} \frac{\eta(b, a)}{2^{1+1/q}} \left[|f'(a)|^q + \frac{1-\lambda}{\lambda} |f'(b)|^q \right] \\
& \quad \times \left\{ \frac{2\sqrt{\pi}\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha+2)} - \frac{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha+2)} - 4B_{\frac{1}{2}}(\alpha + \frac{3}{2}, \frac{1}{2}) \right. \\
& \quad \left. + \frac{2^{-\alpha}(- (4\alpha^2 + 18\alpha + 19) {}_2F_1(1, \alpha+2; \frac{1}{2}; \frac{1}{2}) - 2(\alpha+2) {}_2F_1(1, \alpha+2; -\frac{1}{2}; \frac{1}{2}))}{4\alpha^2 + 8\alpha + 3} \right. \\
& \quad \left. + \frac{2^{-\alpha}(-\alpha + 2^{\alpha+\frac{1}{2}} {}_2F_1(-\frac{1}{2}, \frac{1}{2} - \alpha; \frac{1}{2}; \frac{1}{2}) - 1)}{\alpha(\alpha+1)} \right\}^{1/q}.
\end{aligned}$$

The proof is done. \square

Remark 8. In Theorem 4, if we choose $\eta(b, a) = b - a$ and $\alpha = 1$, then we have;

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left(\frac{1}{4} \right)^{\frac{q-1}{q}} \frac{b-a}{2^{1+1/q}} \left[|f'(a)|^q + \frac{1-\lambda}{\lambda} |f'(b)|^q \right].$$

Remark 9. In Theorem 4, if we choose $\eta(b, a) = b - a$, $\lambda = 1/2$ and $\alpha = 1$, then we have;

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq 2^{\frac{1}{q}} \frac{b-a}{8} [|f'(a)|^q + |f'(b)|^q].$$

Lemma 6. Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. If $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f'' \in L[a, a + \eta(b, a)]$ then, the following equality holds:

$$(2.3) \quad \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} [J_{a+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b, a))^-}^\alpha f(a)] \right| \\ = \frac{(\eta(b, a))^2}{2(\alpha + 1)} \int_0^1 [1 - (1-t)^{\alpha+1} - t^{\alpha+1}] f''(a + (1-t)\eta(b, a)) dt.$$

Proof. Integrating by part and changing the variable of definite integral yield

$$(2.4) \quad \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right] f''(a + (1-t)\eta(b, a)) dt \\ = - \left. \frac{(1 - (1-t)^{\alpha+1} - t^{\alpha+1}) f'(a + (1-t)\eta(b, a))}{(\alpha + 1)\eta(b, a)} \right|_0^1 \\ + \frac{1}{\eta(b, a)} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(a + (1-t)\eta(b, a)) dt \\ = \frac{1}{\eta(b, a)} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(a + (1-t)\eta(b, a)) dt.$$

Motivated by Lemma 5, then:

$$\frac{1}{\eta(b, a)} \left(\int_0^1 [(1-t)^\alpha - t^\alpha] f'(a + (1-t)\eta(b, a)) dt \right) \\ = \frac{f(a) + f(a + \eta(b, a))}{(\eta(b, a))^2} - \frac{\Gamma(\alpha + 1)}{(\eta(b, a))^{\alpha+2}} [J_{a+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b, a))^-}^\alpha f(a)].$$

By multiplying the both sides of (2.4) by $\frac{(\eta(b, a))^2}{2}$, we have:

$$\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} [J_{a+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b, a))^-}^\alpha f(a)] \right| \\ = \frac{(\eta(b, a))^2}{2} \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right] f''(a + (1-t)\eta(b, a)) dt$$

The proof is done. \square

Remark 10. In Lemma 6, $\eta(b, a) = b - a$. Lemma 6 reduces to Lemma 2;

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \\ &= \frac{(b-a)^2}{2} \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] f''(ta + (1-t)b) dt. \end{aligned}$$

Theorem 5. Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f'' \in L[a, a + \eta(b, a)]$. If $|f''|$ is λ -preinvex function on $[a, a + \eta(b, a)]$ then the following inequality for fractional integrals with $\alpha > 0$ holds:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} [J_{a+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b, a))^-}^\alpha f(a)] \right| \\ & \leq \frac{(\eta(b, a))^2}{4(\alpha + 1)} \left(\frac{\pi}{2} - \frac{\sqrt{\pi}\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha + 2)} \right) \left\{ |f''(a)| + \left(\frac{1-\lambda}{\lambda} \right) |f''(b)| \right\}. \end{aligned}$$

Proof. By using Definition 7 and Lemma 6, we have:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} [J_{a+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b, a))^-}^\alpha f(a)] \right| \\ & \leq \frac{(\eta(b, a))^2}{2} \int_0^1 \left| \frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right| |f''(a + (1-t)\eta(b, a))| dt \\ & \leq \frac{(\eta(b, a))^2}{2(\alpha+1)} \int_0^1 |1 - (1-t)^{\alpha+1} - t^{\alpha+1}| \left(\frac{\sqrt{t}}{2\sqrt{1-t}} |f''(a)| + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f''(b)| \right) dt \\ & \leq \frac{(\eta(b, a))^2}{2(\alpha+1)} \left\{ \frac{|f''(a)|}{2} \int_0^1 (1 - (1-t)^{\alpha+1} - t^{\alpha+1}) \frac{\sqrt{t}}{\sqrt{1-t}} dt \right. \\ & \quad \left. + \frac{1-\lambda}{\lambda} \frac{|f''(b)|}{2} \int_0^1 (1 - (1-t)^{\alpha+1} - t^{\alpha+1}) \frac{\sqrt{1-t}}{\sqrt{t}} dt \right\} \\ & \leq \frac{(\eta(b, a))^2}{4(\alpha+1)} \left(\frac{\pi}{2} - \frac{\sqrt{\pi}\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha + 2)} \right) \left\{ |f''(a)| + \left(\frac{1-\lambda}{\lambda} \right) |f''(b)| \right\}. \end{aligned}$$

The proof is done. \square

Remark 11. In Theorem 5, if we take $\eta(b, a) = b - a$, we have;

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)^2}{4(\alpha+1)} \left(\frac{\pi}{2} - \frac{\sqrt{\pi}\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha + 2)} \right) \left\{ |f''(a)| + \left(\frac{1-\lambda}{\lambda} \right) |f''(b)| \right\}. \end{aligned}$$

Remark 12. In Theorem 5, if we take $\eta(b, a) = b - a$ and $\alpha = 1$, we have;

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\pi(b-a)^2}{64} \left\{ |f''(a)| + \left(\frac{1-\lambda}{\lambda} \right) |f''(b)| \right\}.$$

Remark 13. In Theorem 5, if we take $\eta(b, a) = b - a$, $\lambda = \frac{1}{2}$ and $\alpha = 1$, we have;

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\pi(b-a)^2}{64} \{|f''(a)| + |f''(b)|\}.$$

Theorem 6. Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f'' \in L[a, a + \eta(b, a)]$. If $|f''|^q$ is λ -preinvex function on $[a, a + \eta(b, a)]$ for some fixed $q > 1$ then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} [J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a + \eta(b, a))^-}^\alpha f(a)] \right| \\ & \leq \frac{(\eta(b, a))^2}{2(\alpha + 1)} (1 - 2^{-\alpha}) \frac{\pi}{4} \left(|f''(a)|^q + \frac{1-\lambda}{\lambda} |f''(b)|^q \right)^{\frac{1}{q}} \end{aligned}$$

where $\alpha \in [0, 1]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Definition 7, Lemma 6 and Hölder's inequality we have:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} [J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a + \eta(b, a))^-}^\alpha f(a)] \right| \\ & \leq \frac{(\eta(b, a))^2}{2} \int_0^1 \left| \frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right| |f''(a + (1-t)\eta(b, a))| dt \\ & \leq \frac{(\eta(b, a))^2}{2(\alpha + 1)} \left(\int_0^1 [1 - (1-t)^{\alpha+1} - t^{\alpha+1}]^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(a + (1-t)\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(\eta(b, a))^2}{2(\alpha + 1)} \left(\int_0^1 [1 - 2^{-\alpha}]^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left(\frac{\sqrt{t}}{2\sqrt{1-t}} |f''(a)|^q + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f''(b)|^q \right) dt \right)^{\frac{1}{q}} \\ & \leq \frac{(\eta(b, a))^2}{2(\alpha + 1)} (1 - 2^{-\alpha}) \frac{\pi}{4} \left(|f''(a)|^q + \frac{1-\lambda}{\lambda} |f''(b)|^q \right)^{\frac{1}{q}}. \end{aligned}$$

The proof is done. \square

Remark 14. In Theorem 6, if we take $\eta(b, a) = b - a$, we have;

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)^2}{2(\alpha + 1)} (1 - 2^{-\alpha}) \frac{\pi}{4} \left(|f''(a)|^q + \frac{1-\lambda}{\lambda} |f''(b)|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Remark 15. In Theorem 6, if we take $\eta(b, a) = b - a$ and $\alpha = 1$, we have;

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{8} \frac{\pi}{4} \left(|f''(a)|^q + \frac{1-\lambda}{\lambda} |f''(b)|^q \right)^{\frac{1}{q}}.$$

Remark 16. In Theorem 6, if we take $\eta(b, a) = b - a$, $\lambda = \frac{1}{2}$ and $\alpha = 1$, we have;

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{8} \frac{\pi}{4} (|f''(a)|^q + |f''(b)|^q)^{\frac{1}{q}}.$$

Theorem 7. Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f'' \in L[a, a + \eta(b, a)]$. If $|f''|^q$ is λ -preinvex function on $[a, a + \eta(b, a)]$ for some fixed $q > 1$ then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} \left[J_{a+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b, a))^-}^\alpha f(a) \right] \right| \\ & \leq \frac{(\eta(b, a))^2}{2(\alpha + 1)} \left(\frac{\alpha}{\alpha + 2} \right)^{1-\frac{1}{q}} \left(\frac{\pi}{2} - \frac{\sqrt{\pi}\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha + 2)} \right)^{1/q} \left(\frac{|f''(a)|^q}{2} + \frac{1-\lambda}{\lambda} \frac{|f''(b)|^q}{2} \right)^{1/q}. \end{aligned}$$

where $\alpha \in [0, 1]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Definition 7, Lemma 6 and Power Mean's inequality, we have:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2(\eta(b, a))^\alpha} \left[J_{a+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b, a))^-}^\alpha f(a) \right] \right| \\ & \leq \frac{(\eta(b, a))^2}{2} \int_0^1 \left| \frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right| |f''(a + (1-t)\eta(b, a))| dt \\ & \leq \frac{(\eta(b, a))^2}{2(\alpha + 1)} \left(\int_0^1 \left| 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right| dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\int_0^1 \left| 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right| |f''(a + (1-t)\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(\eta(b, a))^2}{2(\alpha + 1)} \left(\int_0^1 \left[1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\int_0^1 \left[1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] \left(\frac{\sqrt{t}}{2\sqrt{1-t}} |f''(a)|^q + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} |f''(b)|^q \right) dt \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\eta(b,a))^2}{2(\alpha+1)} \left(\frac{\alpha}{\alpha+2} \right)^{1-\frac{1}{q}} \times \left(\frac{|f''(a)|^q}{2} \int_0^1 \left[1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] \frac{\sqrt{t}}{\sqrt{1-t}} dt \right. \\
&\quad \left. + \left(\frac{1-\lambda}{\lambda} \right) \frac{|f''(b)|^q}{2} \int_0^1 \left[1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] \frac{\sqrt{1-t}}{\sqrt{t}} dt \right)^{\frac{1}{q}} \\
&\leq \frac{(\eta(b,a))^2}{2(\alpha+1)} \left(\frac{\alpha}{\alpha+2} \right)^{1-\frac{1}{q}} \left(\frac{\pi}{2} - \frac{\sqrt{\pi}\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+2)} \right)^{1/q} \left(\frac{|f''(a)|^q}{2} + \frac{1-\lambda}{\lambda} \frac{|f''(b)|^q}{2} \right)^{1/q}.
\end{aligned}$$

The proof is done. \square

Remark 17. In Theorem 7, if we take $\eta(b, a) = b - a$, we have;

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\
&\leq \frac{(b-a)^2}{2(\alpha+1)} \left(\frac{\alpha}{\alpha+2} \right)^{1-\frac{1}{q}} \left(\frac{\pi}{2} - \frac{\sqrt{\pi}\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+2)} \right)^{1/q} \left(\frac{|f''(a)|^q}{2} + \frac{1-\lambda}{\lambda} \frac{|f''(b)|^q}{2} \right)^{1/q}.
\end{aligned}$$

Remark 18. In Theorem 7, if we take $\eta(b, a) = b - a$ and $\alpha = 1$, we have;

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{4} \left(\frac{1}{3} \right)^{1-\frac{1}{q}} \left(\frac{\pi}{8} \right)^{1/q} \left(\frac{|f''(a)|^q}{2} + \frac{1-\lambda}{\lambda} \frac{|f''(b)|^q}{2} \right)^{1/q}.$$

Remark 19. In Theorem 7, if we take $\eta(b, a) = b - a$, $\lambda = \frac{1}{2}$ and $\alpha = 1$, we have;

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{4} \left(\frac{1}{3} \right)^{1-\frac{1}{q}} \left(\frac{\pi}{8} \right)^{1/q} \left(\frac{|f''(a)|^q}{2} + \frac{|f''(b)|^q}{2} \right)^{1/q}.$$

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